Eigenvalues, Eigenvectors, and Eigenspaces

Def^n: Let $L: V \to V$ be a linear operator on vector space V. A nonzero vector $v \in V$ is an <u>eigenvector</u> with <u>eigenvalue</u> λ when $L(v) = \lambda v$.

Recall that an $n \times n$ matrix determines a linear transformation $L_M : \mathbb{R}^n \to \mathbb{R}^n$ where $\mathrm{Rep}_{\mathcal{E}_n, \mathcal{E}_n}(L_M) = M$. When we discuss the eigenvalues or eigenvectors of a matrix, we mean the corresponding object for the transformation L_M. Note that the correspondence between $n \times n$ matrices and linear operators on $\mathbb{R}^n$ allows us to work primarily with matrices from now on.

Ex: Let $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Noting that

$$M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$ we see that

$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of M with eigenvalue $\lambda = 2$. $\boxed{//}$

Note that each eigenvalue of M yields a subspace of $\mathbb{R}^n$.

Prop: Let λ be a scalar and $L: V \to V$ a linear operator. The set $V_\lambda := \{ u \in V : L(u) = \lambda u \}$ is a subspace of V.

pf: We apply the subspace test. In particular, given two elements $u, v \in V_\lambda$ and scalar a, we compute

$$\begin{aligned}
L(u + av) &= L(u) + aL(v) && \text{(by linearity of } L\text{)} \\
&= \lambda u + a(\lambda v) && \text{(definition of } V_\lambda\text{)} \\
&= \lambda u + (a\lambda) v && \text{(vector space axiom)} \\
&= \lambda u + (\lambda a) v && \text{(commute multiplication)} \\
&= \lambda u + \lambda (av) && \text{(vector space axiom)} \\
&= \lambda (u + av) && \text{(scalar distribution)}
\end{aligned}$$

Hence $L(u + av) = \lambda(u + av)$ yields $u + av \in V_\lambda$. Note also $L(0_V) = 0_V = \lambda \cdot 0_V$, so $0_V \in V_\lambda \neq \emptyset$. Hence $V_\lambda \leq V$ as desired. $\boxed{//}$

Def^n: The spaces $V_\lambda := \{u \in V : L(u) = \lambda u\}$ are eigenspaces.

Observation: If $v \in V_\lambda \cap V_\mu$ and $v \neq 0$, then $\lambda v = L(v) = \mu v$. Thus $(\lambda - \mu) v = \lambda v - \mu v = 0_v$, so we have $\lambda - \mu = 0$, i.e. $\lambda = \mu$. In particular, eigenspaces of distinct eigenvalues have only the zero vector in common ☺

At this point, we've seen an example and played with some theory. But how do we compute eigenvalues and eigenspaces?

If v is an eigenvector of M with eigenvalue λ, then $Mv = \lambda v$. Subtracting λv we obtain

$$0_v = Mv - \lambda v = Mv - \lambda I v = (M - \lambda I) v.$$

From this we've learned two new facts.

① If λ is an eigenvalue of M, then $M - \lambda I$ is singular.

② Every eigenvector of M with eigenvalue λ is in $\text{null}(M - \lambda I)$.

For the moment let's focus on condition ①. The matrix $M - \lambda I$ is singular if and only if $\det(M - \lambda I) = 0$. This simple observation leads us to make a definition.

Def^n: The characteristic polynomial of an $n \times n$ matrix M is $P_M(\lambda) := \det(M - \lambda I)$ where λ is a variable.

Now we formalize our observation from above.

Prop: Let M be a matrix. A scalar λ is an eigenvalue of M if and only λ is a root of P_M.

Point: To compute eigenvalues, we need only compute roots of P_M ☺

Ex: Compute the eigenvalues of $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Sol: First we compute the characteristic polynomial of M.

$$P_M(\lambda) = \det(M - \lambda I) = \det\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

Cofactor expansion on row 1

$$= (1-\lambda) \det \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} - \det \begin{bmatrix} 1 & 1 \\ 0 & 1-\lambda \end{bmatrix} + 0$$

2x2 determinant formula

$$= (1-\lambda)\big(-\lambda(1-\lambda) - 1\big) - \big((1-\lambda) - 0\big)$$

basic algebra techniques

$$\left.\begin{aligned} &= -(1-\lambda)(1 + \lambda - \lambda^2) - (1-\lambda) \\ &= -(1-\lambda)(1 + \lambda - \lambda^2 + 1) \\ &= +(1-\lambda)\big(+(\lambda^2 - \lambda - 2)\big) \\ &= (1-\lambda)(\lambda - 2)(\lambda + 1) = -(\lambda+1)(\lambda-1)(\lambda-2) \end{aligned}\right.$$

Hence $P_M(\lambda) = -(\lambda+1)(\lambda-1)(\lambda-2)$ is the characteristic polynomial.

Now we compute the eigenvalues of M by solving $P_M(\lambda) = 0$:

$$P_M(\lambda) = 0 \iff -(\lambda+1)(\lambda-1)(\lambda-2) = 0$$

$$\iff \lambda + 1 = 0 \quad \text{OR} \quad \lambda - 1 = 0 \quad \text{OR} \quad \lambda - 2 = 0$$

$$\iff \lambda = -1 \quad \text{OR} \quad \lambda = 1 \quad \text{OR} \quad \lambda = 2$$

Hence M has eigenvalues $\lambda = -1$, $\lambda = 1$, and $\lambda = 2$. ▨

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has $P_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2$,

so $\lambda = 1$ is the only eigenvalue of A.

Ex: $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ has characteristic polynomial

$$p_B(\lambda) = \det(B - \lambda I) = \det\begin{bmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 2.$$

Hence we compute eigenvalues as follows:

$$\begin{aligned} p_B(\lambda) = 0 &\iff (1-\lambda)^2 - 2 = 0 \\ &\iff (1-\lambda)^2 = 2 \\ &\iff 1 - \lambda = \pm\sqrt{2} \\ &\iff -\lambda = -1 \pm \sqrt{2} \\ &\iff \lambda = 1 \pm \sqrt{2} \end{aligned}$$

Thus B has eigenvalues $\lambda = 1 \pm \sqrt{2}$. ▨

Ex: $C = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$ has characteristic polynomial

$$\begin{aligned} p_C(\lambda) &= \det(C - \lambda I) \\ &= \det\begin{bmatrix} 1-\lambda & 3 \\ -1 & 2-\lambda \end{bmatrix} \\ &= (1-\lambda)(2-\lambda) - (-1)(3) \\ &= 2 - 3\lambda + \lambda^2 + 3 \\ &= \lambda^2 - 3\lambda + 5 \\ &= \underbrace{\left(\lambda^2 - 2\left(\tfrac{3}{2}\right)\lambda + \left(\tfrac{3}{2}\right)^2\right)} + \left(5 - \left(\tfrac{3}{2}\right)^2\right) \\ &= \left(\lambda - \tfrac{3}{2}\right)^2 + \left(5 - \tfrac{9}{4}\right) \\ &= \left(\lambda - \tfrac{3}{2}\right)^2 + \tfrac{11}{4} \end{aligned}$$

2x2 determinant formula

Simplify

"Complete the square"

Hence $p_C(\lambda) = \left(\lambda - \frac{3}{2}\right)^2 + \frac{11}{4}$, which has complex roots!

Indeed, the eigenvalues of C are $\lambda = \frac{3}{2} \pm \frac{\sqrt{11}}{2} i$. ▨

NB: The last example indicates eigenvalues can be complex!

In the background we're actually working with $L_A : \mathbb{C}^2 \to \mathbb{C}^2$,

and V_λ is a complex vector space now :)

At this point we know how to compute eigenvalues via the characteristic polynomial. But what about eigenvectors and eigenspaces? For that we formalize observation ② from earlier.

<u>Prop</u>: Let M be an $n \times n$ matrix with eigenvalue λ. The eigenspace of M associated to λ is $V_\lambda = \text{null}(M - \lambda I)$.

<u>Point</u>: To calculate the eigenspaces of M we must

ⓐ Compute $p_M(\lambda)$.

ⓑ solve $p_M(\lambda) = 0$ for eigenvalues.

ⓒ For each eigenvalue λ compute $\text{null}(M - \lambda I)$.

<u>Ex</u>: Let $M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then the characteristic polynomial

$$p_M(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 1 = \lambda(\lambda - 2).$$

Thus M has eigenvalues $\lambda = 0$ and $\lambda = 2$. We must now compute eigenspaces separately via $V_\lambda = \text{null}(M - \lambda I)$.

<u>$\lambda = 0$</u>: $M - 0I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has $\text{RREF}(M - 0I) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$,

So $\begin{bmatrix} x \\ y \end{bmatrix} \in \text{null}(M - 0I) \iff x + y = 0 \iff x = -y$.

Hence $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $V_0 = \text{null}(M - 0I)$.

<u>$\lambda = 2$</u>: $M - 2I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ has $\text{RREF}(M - 2I) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$,

so $\begin{bmatrix} x \\ y \end{bmatrix} \in \text{null}(M - 2I) \iff x - y = 0 \iff x = y$

Hence $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $V_2 = \text{null}(M - 2I)$.

Thus $V_0 = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and $V_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

<u>Ex</u>: Compute the eigenspaces of $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

<u>Sol</u>: Earlier we computed eigenvalues $\lambda = -1, 1, 2$.

<u>$\lambda = -1$</u>: $\text{RREF}(M+I) = \text{RREF}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{null}(M+I) \iff \begin{cases} x = 0 \\ y + z = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = -t \\ z = t \end{cases}$ yields $V_{-1} = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

<u>$\lambda = 1$</u>: $\text{RREF}(M-I) = \text{RREF}\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{null}(M-I) \iff \begin{cases} x + z = 0 \\ y = 0 \end{cases} \iff \begin{cases} x = -t \\ y = 0 \\ z = t \end{cases}$ yields $V_1 = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

<u>$\lambda = 2$</u>: $\text{RREF}(M-2I) = \text{RREF}\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{null}(M-2I) \iff \begin{cases} x - z = 0 \\ y - z = 0 \end{cases} \iff \begin{cases} x = t \\ y = t \\ z = t \end{cases}$ yields $V_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

This finishes the computation of eigenspaces of M. ☒

<u>Ex</u>: Compute the eigenspaces of $F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

<u>Sol</u>: Characteristic polynomial $p_F(\lambda) = \det\begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = -\lambda(1-\lambda) - 1 \cdot 1 = \lambda^2 - \lambda - 1$

has roots $\lambda = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \dfrac{1 \pm \sqrt{5}}{2}$ by the quadratic formula.

Hence we compute the eigenspaces for these eigenvalues below.

<u>$\lambda = \frac{1+\sqrt{5}}{2}$</u>: We compute an echelon form of $F - \lambda I$:

$$\begin{bmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{1+\sqrt{5}}{2} \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1-\sqrt{5} & 2 \\ 2 & 1-\sqrt{5} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1-\sqrt{5} \\ 0 & 0 \end{bmatrix}$$

Hence $\begin{bmatrix} x \\ y \end{bmatrix} \in \text{null}(F - \lambda I) \iff 2x + (1-\sqrt{5})y = 0$

$\iff \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1-\sqrt{5} \\ -2 \end{bmatrix}$ some t

We thus have $V_{\frac{1+\sqrt{5}}{2}} = \text{span}\left\{ \begin{bmatrix} 1-\sqrt{5} \\ -2 \end{bmatrix} \right\}$.

$\underline{\lambda = \frac{1-\sqrt{5}}{2}}$: We compute an echelon form for $F - \lambda I$:

$$\begin{bmatrix} -\frac{1-\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{1-\sqrt{5}}{2} \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1+\sqrt{5} & 2 \\ 2 & 1+\sqrt{5} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1+\sqrt{5} \\ 0 & 0 \end{bmatrix}$$

Hence $\begin{bmatrix} x \\ y \end{bmatrix} \in \text{null}(F - \lambda I) \iff 2x + (1+\sqrt{5})y = 0$

$$\iff \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1+\sqrt{5} \\ -2 \end{bmatrix} \quad \text{some } t$$

Thus we have $V_{\frac{1-\sqrt{5}}{2}} = \text{Span}\left\{ \begin{bmatrix} 1+\sqrt{5} \\ -2 \end{bmatrix} \right\}$. //

$\underline{\text{Ex}}$: Compute the eigenspaces of $M = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

$\underline{\text{Sol}}$: Characteristic polynomial computation yields

$$p_M(\lambda) = \det \begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - (-1) = (\lambda-2)^2 + 1$$

which has roots $\lambda = 2 \pm i$, two complex eigenvalues.

$\underline{\lambda = 2+i}$: $\text{RREF}(M - (2+i)I) = \text{RREF}\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$,

So $\begin{bmatrix} x \\ y \end{bmatrix} \in \text{null}(M - \lambda I) \iff x - iy = 0 \iff \begin{cases} x = it \\ y = t \end{cases}$

and $V_{2+i} = \text{Span}\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$ as a complex vector space.

$\underline{\lambda = 2-i}$: $\text{RREF}(M - (2-i)I) = \text{RREF}\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$,

So $\begin{bmatrix} x \\ y \end{bmatrix} \in \text{null}(M - \lambda I) \iff x + iy = 0 \iff \begin{cases} x = -it \\ y = t \end{cases}$

and $V_{2-i} = \text{Span}\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$ as a complex vector space.

$\underline{\text{NB}}$: The previous examples had all eigenvalues distinct, so this was somewhat special. Indeed, the next few examples are more generic...

Ex: Compute the eigenspaces of $M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

Sol:
$$\begin{aligned} p_M(\lambda) &= \det(M - \lambda I) \\ &= \det\begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 3-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)\det\begin{bmatrix} 3-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} - 0 + 2\det\begin{bmatrix} 0 & 3-\lambda \\ 2 & 0 \end{bmatrix} \\ &= (1-\lambda)\big((3-\lambda)(1-\lambda) - 0\big) + 2\big(0 - 2(3-\lambda)\big) \\ &= (3-\lambda)\big((1-\lambda)^2 - 4\big) \\ &= -(\lambda-3)\big((\lambda-1)^2 - 2^2\big) \\ &= -(\lambda-3)(\lambda-3)(\lambda+1) \\ &= -(\lambda+1)(\lambda-3)^2 \end{aligned}$$

$\therefore$ have eigenvalues $\lambda = -1$, $\lambda = 3$.

$\lambda = -1$: $\text{RREF}(M+I) = \text{RREF}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{null}(M+I) \iff \begin{cases} x + z = 0 \\ y = 0 \end{cases} \iff \begin{cases} x = -t \\ y = 0 \\ z = t \end{cases}$

yields $V_{-1} = \text{null}(M+I) = \text{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

$\lambda = 3$: $\text{RREF}(M-3I) = \text{RREF}\begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

So $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{null}(M-3I) \iff x - z = 0 \iff \begin{cases} x = -t \\ y = s \\ z = t \end{cases} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

some s,t

Hence $V_3 = \text{null}(M-3I) = \text{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

In closing note $\dim(V_{-1}) = 1$ while $\dim(V_3) = 2$.

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Ex: Compute eigenspaces of $M = \begin{bmatrix} \pi & 1 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{bmatrix}$

Sol: $p_M(\lambda) = \det(M - \lambda I) = \det \begin{bmatrix} \pi-\lambda & 1 & 0 \\ 0 & \pi-\lambda & 0 \\ 0 & 0 & \pi-\lambda \end{bmatrix} = (\pi - \lambda)^3.$

Hence we have one eigenspace, for eigenvalue $\lambda = \pi$.

$\lambda = \pi$: $\text{RREF}(M - \pi I) = \text{RREF}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{null}(M - \pi I) \iff y = 0 \iff \begin{cases} x = s \\ y = 0 \\ z = t \end{cases} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s e_1 + t e_3$

Hence $V_\pi = \text{span}\{e_1, e_3\}$. $\boxed{//}$

Note that the dimensions of the eigenspaces were somewhat off-the-walls in the previous few examples. Indeed, we will want to study this somewhat closely for what is to come.

To begin, let's have a definition.

Def^n: Let α be an eigenvalue of M.

① The algebraic multiplicity of α is the power of $(\lambda - \alpha)$ present in the factorization of $p_M(\lambda)$.

② The geometric multiplicity of α is the dimension of V_α.

First we make a simple observation.

Prop: Let α be an eigenvalue of M. The geometric multiplicity of α is at least 1 and at most the algebraic multiplicity of α.

Q: Why care?

A: Before we saw $V_\alpha \cap V_\beta = \{0_V\}$ unless $\alpha = \beta$. This implies that if $B_\alpha \subseteq V_\alpha$ and $B_\beta \subseteq V_\beta$ are bases, then $B_\alpha \cup B_\beta$ is independent in V. As such, geometric multiplicity will tell us if V has a basis of eigenvectors...

Prop: Let M be an $n \times n$ matrix.

① the degree of $P_M(\lambda)$ is n.

② $\mathbb{R}^n$ has a basis of eigenvectors of M if and only if the geometric multiplicity of every eigenvalue is equal to its algebraic multiplicity.

Recall that matrices A and B are similar when there is an invertible matrix P such that $B = P^{-1}AP$. We say matrix M is diagonalizable when there is a diagonal matrix D which is similar to M.

Prop (Diagonalizability Criterion) Let M be an $n \times n$ matrix. The following are equivalent.

① M is diagonalizable.

② Each eigenvalue of M has equal algebraic and geometric multiplicity.

③ $\mathbb{R}^n$ has a basis $\mathcal{B}$ in which every vector of $\mathcal{B}$ is an eigenvector of M.

Construction: to diagonalize M:

① Compute $P_M(\lambda)$ and eigenvalues of M.

② Compute a basis of the eigenspace of each eigenvalue.

↳ If $\dim(V_\alpha)$ is less than the algebraic multiplicity of α, then STOP (it's not possible)

③ Consider $\mathcal{B} = \mathcal{B}_{\lambda_1} \cup \mathcal{B}_{\lambda_2} \cup \cdots \cup \mathcal{B}_{\lambda_k}$ where $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ are the eigenvalues of M and $\mathcal{B}_{\lambda_i}$ is a basis of V_{λ_i} for all i.

④ Let $P = \mathrm{Rep}_{\mathcal{B},\mathcal{E}_n}(id)$, so $P^{-1} = \mathrm{Rep}_{\mathcal{E}_n,\mathcal{B}}(id)$.

Then $D = P^{-1}MP$ is diagonal (if step 2 did not fail).